

# BIRATIONAL ASPECTS OF THE GEOMETRY OF VARIETIES OF SUM OF POWERS

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**ABSTRACT.** Varieties of Sums of Powers describe the additive decompositions of an homogeneous polynomial into powers of linear forms. Despite their long history, going back to Sylvester and Hilbert, few of them are known for special degrees and number of variables. In this paper we aim to understand a general birational behaviour of *VSP*, if any. To do this we birationally embed these varieties into Grassmannians and prove the rationality, unirationality or rational connectedness of many of those in arbitrary degrees and number of variables.

## INTRODUCTION

In 1770 Edward Waring stated that every integer is a sum of at most 9 positive cubes, later on Jacobi and others considered the problem to find all the decompositions of a given number into a number of cubes, [Di]. Since then many problems related to additive decomposition are named after Waring. The set up we are interested in is that of homogeneous polynomials. Let  $F \in k[x_0, \dots, x_n]_d$  be a general homogeneous polynomial of degree  $d$ . The additive decomposition we are looking for is

$$F = L_1^d + \dots + L_h^d,$$

where  $L_i \in k[x_0, \dots, x_n]_1$  are linear forms. The problem is very classical, the first results are due to *Sylvester*, [Sy] and then to *Hilbert*, [Hi], *Richmond*, [Ri], *Palatini*, [Pa], and many others. In the old times the attention was essentially devoted to study the cases in which the above decomposition is unique. This gives a canonical form to general homogeneous polynomials of a particular degree and number of variables. As widely expected the canonical form very seldom exists, see [Me1] [Me2]. In the remaining cases one should try to understand the set of decompositions of a given general polynomial. A compactification of this is usually called the *Variety of Sums of Powers* (*VSP* for short), see Definition 1.4 for the precise statement. The interest in these special varieties increased greatly after *Mukai*, [Mu1], gave a description of the Fano 3-fold  $V_{22}$  as *VSP* of quartic polynomials in three variables. Since then different authors exploited the area and generalize Mukai's techniques to other polynomials, [DK], [RS], [IR1], [IR2], [TZ], see [Do] for a very nice survey. All these works address special values of degree and variables and give a biregular description of special *VSP*'s. This is done studying

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the natural compactification  $VSP(F, h)$  of the additive decompositions into the Hilbert scheme of  $(\mathbb{P}^n)^*$ , see Section 1 for the details. The known cases are not many and, to the best of our knowledge, this is the state of the art

d	n	h	$VSP(F_d, h)$	Reference
2h-1	1	h	1 point	Sylvester [Sy]
2	2	3	quintic Fano threefold	Mukai [Mu1]
3	2	4	$\mathbb{P}^2$	Dolgachev and Kanev [DK]
4	2	6	Fano 3-fold $V_{22}$	Mukai [Mu1]
5	2	7	1 point	Hilbert, [Hi], Richmond, [Ri]
				Palatini, [Pa]
6	2	10	K3 surface of genus 20	Mukai [Mu2]
7	2	12	5 points	Dixon and Stuart [Dx]
8	2	15	16 points	Mukai [Mu2]
3	3	5	1 point	Sylvester's Pentahedral Theorem [Sy]
3	4	8	$\mathcal{W}$	Ranestad and Schreier [RS]
3	5	10	$\mathcal{S}$	Iliev and Ranestad [IR1]

where  $\mathcal{W}$  is a fivefold and is the variety of lines in the fivefold linear complete intersection  $\mathbb{P}^{10} \cap \mathbb{O}\mathbb{G}(5, 10) \subseteq \mathbb{P}^{15}$  of the ten-dimensional orthogonal Grassmanian  $\mathbb{O}\mathbb{G}(5, 10)$ , and  $\mathcal{S}$  is a smooth symplectic fourfold obtained as a deformation of the Hilbert square of a polarized K3 surface of genus eight.

In this paper we aim to understand a general birational behaviour of  $VSP$ , if any. To do this we prefer to adopt a different compactification. This is probably less efficient than the usual one to study the biregular nature of these objects but is very suitable for birational purposes.

Let  $F \in k[x_0, \dots, x_n]_d$  be a general homogeneous polynomial of degree  $d$  and  $V = V_{d,n} \subset \mathbb{P}^N = \text{Proj}(k[x_0, \dots, x_n]_d)$  the Veronese variety. A general additive decomposition into  $h$  linear factors

$$F = \sum_{i=1}^h \lambda_i L_i^d$$

is associated to an  $h$ -secant linear space of dimension  $h-1$  to the Veronese  $V \subset \mathbb{P}^N$ . In this way we can embed a general additive decomposition into either  $\mathbb{G}(h-1, N)$ , or  $\mathbb{G}(h-2, N-1)$ , and consider the closure there. This compactification is usually more singular than the one into the Hilbert scheme and meaningful only for  $h < N-n$ , see Remark 1.8 for a brief comparison. Nonetheless it allows us to use projective techniques in a wider context. In this way we are able to give several interesting result about the birational nature of  $VSP$ 's.

**Theorem 1.** *Assume that  $F$  is a general quadratic polynomial in  $n+1$  variables. Then the irreducible components of  $VSP(F, h)$  are unirational for any  $h$  and rational for  $h = n+1$ .*

Theorem 1 cannot be extended to higher degrees. Think for instance to the mentioned examples of either *Mukai* or *Iliev* and *Ranestad*. On the other hand we believe that Rational Connectedness is the general pattern for this class of varieties. In this direction our main result is the rational connectedness of infinitely many  $VSP$  with arbitrarily high degree and number of variable. Having in mind that  $VSP(F, h)$  are non empty only if  $h \geq \binom{d+n}{n}/(n+1)$  we can prove the following.

**Theorem 2.** Assume that for some positive integer  $k < n$  the number  $\frac{\binom{d+n}{n}-1}{k+1}$  is an integer. Then the irreducible components of  $VSP(F, h)$  are Rationally Connected for  $F \in k[x_0, \dots, x_n]_d$  general and  $h \geq \frac{\binom{n+d}{n}-1}{k+1}$ .

The common kernel of these theorems is Theorem 3.2 where, under suitable assumption, we connect  $VSP(F, h)$  with chains of  $VSP(F, h-1)$ . In this way we reduce the rational connectedness computations to special values of  $h$ . This new approach allows us also to recover and reinterpret known classification results, see section 2, and generalize a Theorem of Sylvester, see section 4.

## 1. NOTATION AND PRELIMINARIES

We work over the complex field. We mainly follow notation and definitions of [Do]. Let  $V$  be a vector space of dimension  $n+1$  and let  $\mathbb{P}(V) = \mathbb{P}^n$  the corresponding projective space. For any finite set of points  $\{p_1, \dots, p_h\} \subseteq \mathbb{P}^n$  we consider the linear space of homogeneous forms  $F$  of degree  $d$  on  $\mathbb{P}^n$  such that  $(F=0)$  contains the points  $p_1, \dots, p_h$ , and we denote it by

$$L_d(p_1, \dots, p_h) = \{F \in k[x_0, \dots, x_n]_d \mid p_i \in (F=0) \forall 1 \leq i \leq h\}.$$

**Definition 1.1.** An unordered set of points  $\{[L_1], \dots, [L_h]\} \subseteq \mathbb{P}V^*$  is a *polar  $h$ -polyhedron* of  $F \in k[x_0, \dots, x_n]_d$  if

$$F = \lambda_1 L_1^d + \dots + \lambda_h L_h^d,$$

for some nonzero scalars  $\lambda_1, \dots, \lambda_h \in k$  and moreover the  $L_i^d$  are linearly independent in  $k[x_0, \dots, x_n]_d$ .

We briefly introduce the concept of Apolar form to a given homogeneous form to state the connection between the set of  $h$ -polyhedra of  $F$  and the space of Apolar forms of  $F$ .

Fix a system of coordinates  $\{x_0, \dots, x_n\}$  on  $V$  and the dual coordinates  $\{\xi_0, \dots, \xi_n\}$  on  $V^*$ .

Let  $\phi = \phi(\xi_0, \dots, \xi_n)$  be a homogeneous polynomial of degree  $t$  on  $V^*$ . We consider the differential operator

$$D_\phi = \phi(\partial_0, \dots, \partial_n), \text{ with } \partial_i = \frac{\partial}{\partial x_i}.$$

This operator acts on  $\phi$  substituting the variable  $\xi_i$  with the partial derivative  $\partial_i = \frac{\partial}{\partial x_i}$ . For any  $F \in k[x_0, \dots, x_n]_d$  we write

$$\langle \phi, F \rangle = D_\phi(F).$$

We call this pairing the *apolarity pairing*.

In general  $\phi$  is of the form  $\phi(\xi_0, \dots, \xi_n) = \sum_{i_0+\dots+i_n=t} \alpha_{i_0, \dots, i_n} \xi_0^{i_0} \dots \xi_n^{i_n}$  and  $F$  is of the form  $F(x_0, \dots, x_n) = \sum_{j_0+\dots+j_n=d} f_{j_0, \dots, j_n} x_0^{j_0} \dots x_n^{j_n}$ . Then

$$D_\phi(F) = (\sum_{i_0+\dots+i_n=t} \alpha_{i_0, \dots, i_n} \partial_0^{i_0} \dots \partial_n^{i_n})(F).$$

We see that  $F$  is derived  $i_0 + \dots + i_n = t$  times. So we obtain a homogeneous polynomial of degree  $d-t$  on  $V$ .

Fixed  $F \in k[x_0, \dots, x_n]_d$  we have the map

$$ap_F^t : k[\xi_0, \dots, \xi_n]_t \rightarrow k[x_0, \dots, x_n]_{d-t}, \quad \phi \mapsto D_\phi(F).$$

The map  $ap_F^t$  is linear and we can consider the subspace  $Ker(ap_F^t)$  of  $k[\xi_0, \dots, \xi_n]_t$ .

**Definition 1.2.** A homogeneous form  $\phi \in k[\xi_0, \dots, \xi_n]_t$  is called *Apolar* to a homogeneous form  $F \in k[x_0, \dots, x_n]_d$  if  $D_\phi(F) = 0$ , in other words if  $\phi \in \text{Ker}(ap_F^t)$ . The vector subspace of  $k[\xi_0, \dots, \xi_n]_t$  of Apolar forms of degree  $t$  to  $F$  is denoted by  $AP_t(F)$ .

**Lemma 1.3** ([Do]). *The set  $\mathcal{P} = \{[L_1], \dots, [L_h]\}$  is a polar  $h$ -polyhedron of  $F$  if and only if*

$$L_d([L_1], \dots, [L_h]) \subseteq AP_d(F),$$

*and the inclusion is not true if we delete any  $[L_i]$  from  $\mathcal{P}$ .*

The set of all  $h$ -polyhedra of a general polynomial  $F \in k[x_0, \dots, x_n]_d$  is denoted by  $VSP(F, h)^o$ . Via this construction it is easy to embed  $VSP(F, h)^o$  into  $\text{Hilb}_h((\mathbb{P}^n)^*)$ .

**Definition 1.4.** The closure

$$VSP(F, h) := \overline{VSP(F, h)^o} \subseteq \text{Hilb}_h((\mathbb{P}^n)^*)$$

is the *Variety of Sums of Powers* of  $F$ . The points in  $VSP(F, h) \setminus VSP(F, h)^o$  are called generalized polar polyhedra.

Using the smoothness of  $\text{Hilb}_h((\mathbb{P}^n)^*)$ , when  $n = 1, 2$ , one gets the following classical result, see for instance [Do].

**Proposition 1.5.** *In the cases  $n = 1, 2$  for a general polynomial  $F \in k[x_0, \dots, x_n]_d$  the variety  $VSP(F, h)$  is either empty or a smooth variety of dimension*

$$\dim(VSP(F, h)) = h(n+1) - \binom{n+d}{d}.$$

It is important to notice that an additive decomposition of  $F$  induces an additive decomposition of its partial derivatives.

*Remark 1.6* (Partial Derivatives). Let  $\{[L_1], \dots, [L_h]\}$  be a  $h$ -polar polyhedron for the homogeneous polynomial  $F \in k[x_0, \dots, x_n]_d$ . We write

$$F = \lambda_1 L_1^d + \dots + \lambda_h L_h^d.$$

The partial derivatives of  $F$  are homogeneous polynomials of degree  $d-1$  decomposed in  $h$  linear factors

$$\frac{\partial F}{\partial x_i} = \lambda_1 \alpha_{i1} d L_1^{d-1} + \dots + \lambda_h \alpha_{ih} d L_h^{d-1}, \text{ for any } i = 0, \dots, n.$$

Hence, as long as  $h < \binom{d-1+n}{n}$ ,  $VSP(F, h)^o \subseteq VSP(\frac{\partial F}{\partial x_i}, h)^o$ , and taking closures we have

$$VSP(F, h) \subseteq VSP(\frac{\partial F}{\partial x_i}, h).$$

The polynomial  $F$  has  $\binom{n+l}{l}$  partial derivatives of order  $l$ . Clearly these derivatives are homogeneous polynomials of degree  $d-l$  decomposed in  $h$ -linear factors. Then, when  $h < \binom{d-l+n}{n}$ , we have  $VSP(F, h) \subseteq VSP(\frac{\partial^l F}{\partial x_0^{l_0} \dots \partial x_n^{l_n}}, h)$ , where  $l_0 + \dots + l_n = l$ .

As remarked in the introduction we are interested in a different compactification of additive decompositions. Consider the span of the polar polyhedron in the Veronese embedding. We can associate to an  $h$ -polar polyhedron of  $F$  an  $(h-1)$ -plane  $h$ -secant to the Veronese variety  $V_{d,n} \subset \mathbb{P}^N$ . Hence, when  $h < N - n + 1$ , we can embed

$$VSP(F, h)^o \subset \mathbb{G}(h-1, N)$$

as the subvariety formed by the  $(h-1)$ -planes properly secant the Veronese and containing  $[F]$ .

**Definition 1.7.** Let

$$VSP_G(F, h) := \overline{VSP(F, h)}^\circ \subset \mathbb{G}(h-1, N)$$

be the closure in the Grassmannian.

*Remark 1.8.* Note that  $VSP_G(F, h)$  contains limits of  $h$ -secant planes. We expect, in general, that there are no morphisms between the two compactifications but only rational maps. Indeed not all degree  $h$  zero dimensional subschemes of the Veronese variety span a linear space of dimension  $h-1$  and not all limits of  $h$ -secant planes cut a zero dimensional scheme. On the other hand both are clearly true when  $n = 1$  and in this case we have  $VSP(F, h) \cong VSP_G(F, h)$ .

Let us recall, next, the main definitions and results concerning secant varieties. Let  $G_{k-1} = \mathbb{G}(k-1, N)$  be the Grassmannian of  $(k-1)$ -linear spaces in  $\mathbb{P}^N$ . Let  $X \subset \mathbb{P}^N$  be an irreducible variety

$$\Gamma_k(X) \subset X \times \cdots \times X \times G_{k-1},$$

the closure of the graph of

$$\alpha : (X \times \cdots \times X) \setminus \Delta \rightarrow G_{k-1},$$

taking  $(x_1, \dots, x_k)$  to the  $[\langle x_1, \dots, x_k \rangle]$ , for  $k$ -tuple of distinct points. Observe that  $\Gamma_k(X)$  is irreducible of dimension  $kn$ . Let  $\pi_2 : \Gamma_k(X) \rightarrow G_{k-1}$  be the natural projection. Denote by

$$S_k(X) := \pi_2(\Gamma_k(X)) \subset G_{k-1}.$$

Again  $S_k(X)$  is irreducible of dimension  $kn$ . Finally let

$$I_k = \{(x, \Lambda) | x \in \Lambda\} \subset \mathbb{P}^N \times G_{k-1},$$

with natural projections  $\pi_k^X$  and  $\psi_k^X$  onto the factors. Observe that  $\psi_k^X : I_k \rightarrow G_{k-1}$  is a  $\mathbb{P}^{k-1}$ -bundle on  $G_{k-1}$ .

**Definition 1.9.** Let  $X \subset \mathbb{P}^N$  be an irreducible variety. The *abstract  $(k-1)$ -Secant variety* is

$$\text{Sec}_{k-1}(X) := (\psi_k^X)^{-1}(S_k(X)) \subset I_k.$$

While the  *$(k-1)$ -Secant variety* is

$$\text{Sec}_{k-1}(X) := \pi_k^X(\text{Sec}_{k-1}(X)) \subset \mathbb{P}^N.$$

It is immediate that  $\text{Sec}_{k-1}(X)$  is a  $(kn + k - 1)$ -dimensional variety with a  $\mathbb{P}^{k-1}$ -bundle structure on  $S_k(X)$ . One says that  $X$  is  *$(k-1)$ -defective* if

$$\dim \text{Sec}_{k-1}(X) < \min\{\dim \text{Sec}_{k-1}(X), N\}$$

*Remark 1.10.* The definition of abstract and embedded  $(k-1)$ -Secants can be extended to the relative set up of varieties over a scheme  $S$ .

Let  $F \in k[x_0, \dots, x_n]_d$  be a general polynomial. In the notation introduced we have

$$VSP_G(F, h) = \psi_h^{V_{d,n}}((\pi_h^{V_{d,n}})^{-1}([F])) \cong (\pi_h^{V_{d,n}})^{-1}([F]).$$

In what follows we more generally use fibers of  $\pi_k^X$ -maps to study  $VSP$  varieties. For this we slightly abuse the language to state the following

**Definition 1.11.** Let  $X \subset \mathbb{P}^N$  be an irreducible variety, and  $p \in \mathbb{P}^N$  a general point. Then

$$VSP_G^X(h) := (\pi_h^X)^{-1}(p)$$

It is important to stress that  $VSP_G^X(h)$  is not well defined as a variety.

*Remark 1.12 (Partial Derivatives II).* The partial derivatives Remark 1.6 can be strengthened as follows. Let  $[F] \in \mathbb{P}^N$  be a general point. The partial derivatives of  $F$  span a linear space, say  $H_\partial$ , in the corresponding projective space  $\mathbb{P}^{N'}$ . Remark 1.6 tell us that linear spaces associated to polar polyhedra has to contain  $H_\partial$ . In general the opposite is not true but for special values one could be lucky enough to get equality, see for instance Theorems 2.2, and 4.1.

## 2. A NEW VIEWPOINT ON $VSP$

In this section we want to give new insight, and also test our ideas, on well known results about  $VSP(F, h)$ . Let us start with a geometric proof of Hilbert result on the uniqueness of additive decomposition for quintic forms in three variables.

**Theorem 2.1** ([Hi]). *Let  $F \in k[x_0, x_1, x_2]_5$  be a general homogeneous polynomial. Then  $VSP(F, 7)$  is a single point.*

*Proof.* A computation, together with [AH] main result, shows that  $\dim VSP(F, 7) = 0$ . Assume that  $F$  admits two different decompositions, say  $\{[L_1], \dots, [L_7]\}$  and  $\{[l_1], \dots, [l_7]\}$ . Consider the second partial derivatives of  $F$ . Those are six general homogeneous polynomials of degree three. Let  $H_\partial \subseteq \mathbb{P}^9$  be the linear space they generate. Then, by Remark 1.12, we have

$$H_L := \langle [L_1^3], \dots, [L_7^3] \rangle \supset H_\partial \subset \langle [l_1^3], \dots, [l_7^3] \rangle =: H_l$$

The general choice of  $F$  ensures that both  $H_L$  and  $H_l$  intersect the Veronese surface  $V = V_{3,2} \subseteq \mathbb{P}^9$  at 7 distinct points.

Let

$$\pi : \mathbb{P}^9 \dashrightarrow \mathbb{P}^3$$

be the projection from  $H_\partial$ , and  $\overline{V} = \pi(V)$ . Then  $\overline{V}$  is a surface of degree  $\deg(V) = 9$  with two points of multiplicity 7 corresponding to  $\pi(H_L)$  and  $\pi(H_l)$ . This shows that the 7-dimensional linear space  $H := \langle H_L, H_l \rangle$  intersects  $V$  along a curve, say  $\Gamma$ . The construction of  $\Gamma$  yields

$$\deg \Gamma \leq \#(H_L \cap V) = 7.$$

On the other hand  $\deg \Gamma = 3j$  therefore we end up with the following possibilities.

*Case 1* ( $\deg \Gamma = 3$ ). Then  $\Gamma$  is a twisted cubic curve contained in  $H$  and

$$H_l \cdot \Gamma = H_L \cdot \Gamma = 3$$

We may assume that  $H_l \cap \Gamma = \{[l_1^3], [l_2^3], [l_3^3]\}$  and  $H_L \cap \Gamma = \{[L_1^3], [L_2^3], [L_3^3]\}$ . Let  $\Lambda$  be the pencil of hyperplanes containing  $H$ , and  $\nu_3 : \mathbb{P}^2 \rightarrow V$  the Veronese embedding. The linear system  $\nu_3^*(\Lambda|_V)$  is a pencil of conics and therefore  $\#(\text{Bl } \Lambda|_V) \leq 4$ .

To conclude observe that  $\text{Bl } \Lambda|_V \supset H \cap V$ . This force

$$\{[L_4^3], [L_5^3], [L_6^3], [L_7^3]\} = \{[l_4^3], [l_5^3], [l_6^3], [l_7^3]\},$$

and consequently the contradiction  $H_L = H_l$ .

Case 2 ( $\deg \Gamma = 6$ ). Then

$$H_l \cdot \Gamma = H_L \cdot \Gamma = 6$$

We may assume that  $\Gamma \supset \{[L_1^3], \dots, [L_6^3]\} \cup \{[l_1^3], \dots, [l_6^3]\}$ . Let  $\Lambda$  be the pencil of hyperplanes containing  $H$ . Let  $\nu_3 : \mathbb{P}^2 \rightarrow V$  be the Veronese embedding. The linear system  $\nu_3^*(\Lambda|_V)$  is a pencil of lines and therefore  $\#(\text{Bl } \Lambda|_V) = 1$ . This force

$$[L_7^3] = [l_7^3],$$

and consequently the contradiction  $H_L = H_l$ . □

*Sylvester Pentahedral Theorem* can be proved similarly with a slightly more involved argument. Giorgio Ottaviani informed us of a very nice and neat proof using apolarity, for this reason we prefer to skip it. The final  $VSP(F, h)$  we are able to recover is Dolgachev and Kanev's result, [DK] see also [RS]. This proof was actually the starting point of our work.

**Theorem 2.2** ([DK]). *Let  $F \in k[x_0, x_1, x_2]_3$  be a general homogeneous polynomial. Then we have  $VSP(F, 4) \cong \mathbb{P}^2$ .*

*Proof.* Let  $\{[L_1], \dots, [L_4]\}$  be a 4-polar polyhedron of  $F$ . By remark 1.6 it is a 4-polar polyhedron for the partial derivatives  $\frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}$  of  $F$  also. Let

$$H_\partial = \langle \frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2} \rangle \subseteq \mathbb{P}(k[x_0, x_1, x_2]_2) \cong \mathbb{P}^5.$$

The 3-planes, say  $\Pi$ , containing  $H_\partial$  are parametrized by a plane. Moreover the general choice of  $F$  ensures that  $\Pi$  intersects  $V = V_{2,2}$  along a zero dimensional scheme of length 4. This yields a bijective morphism

$$\phi : \mathbb{P}^2 \rightarrow VSP(F, 4), \Pi \mapsto \Pi \cap V.$$

The two varieties are smooth of the same dimension by Proposition 1.5 therefore  $\phi$  is an isomorphism. □

### 3. CHAINS IN $VSP(F, h)$

Let  $F \in k[x_0, \dots, x_n]_d$  be a general homogeneous polynomial of degree  $d$ . Consider a very general additive decomposition

$$F = \sum_1^h \lambda_i L_i^d$$

Let  $p \in VSP(F, h)^o$  the corresponding point. For  $p \in VSP(F, h)^o$  general also the polynomial

$$F - \lambda_1 L_1^d$$

is general and we can view a dense open of  $VSP(F - \lambda_1 L_1^d, h - 1)^o$  as a subvariety of  $VSP(F, h)^o$ . More generally we can consider a flag of subsets

$$VSP(F, h)^o \supset VSP(F - \lambda_1 L_1^d, h - 1)^o \supset \dots \supset VSP(F - \sum_1^r \lambda_i L_i^d, h - r)^o \ni p$$

Passing to the closure in any of the possible compactifications we can cover any Variety of Sum of Powers via  $VSP$  with less addends.

**Convention 3.1.** When working with a general decomposition, say  $\sum_1^h \lambda_i L_i^d$ , we will always tacitly consider the irreducible component of  $VSP(F, h)^o$  containing this general decomposition and keep denoting its compactifications  $VSP(F, h)$ , and  $VSP_G(F, h)$ .

Further note that, when  $VSP(F, h-1)$  is not empty,

$$(1) \quad \text{cod}_{VSP(F, h)^o} VSP(F - \lambda_1 L_1^d, h-1)^o = n+1$$

therefore as long as

$$\dim VSP(F - \lambda_1 L_1^d, h-1)^o \geq n+1$$

we have a well defined intersection theory for these subvarieties in any compactification. Let  $q \in VSP(F - \lambda_1 L_1^d, h-1)^o \subset VSP(F, h)^o$  be a general point then  $q$  represents a decomposition say

$$F = \lambda_1 L_1^d + \sum_2^h \mu_j G_j^d.$$

As long as  $q \neq p$  we may assume that  $\mu_2 G_2 \neq \lambda_i L_i$  for any  $i = 1, \dots, h$ . This means that

$$p \notin VSP(F - \mu_2 G_2^d, h-1)^o.$$

Assume that  $\dim VSP(F - \xi_i M_i^d, h-1) \geq n+1$ . Then by noetherianity we may assume that for a very general point  $p_1$  in  $VSP(F, h)^o$  there is  $M_1$  such that

$$p_1 \in VSP(F - \xi_1 M_1^d, h-1)^o$$

and

$$VSP(F - \xi_i M_i^d, h-1) \cap VSP(F - \mu_2 G_2^d, h-1) \neq \emptyset$$

That is we can connect two very general points of  $VSP(F, h)$  with a chain of varieties of type  $VSP_G^{V_{d,n}}(h-1)$ . We make the above argument explicit in the following Theorem.

**Theorem 3.2.** *Let  $F \in k[x_0, \dots, x_n]_d$  be a general polynomial of degree  $d$ . Assume that  $h \geq \frac{\binom{n+d}{d}}{n+1} + 2$ . Then two very general points of an irreducible component of  $VSP(F, h)$  are joined by a chain (of length at most three) of  $VSP_G^{V_{d,n}}(h-1)$ . Let  $W_i$  be the elements of this chain, and  $q \in W_j \cap W_l$  a general point. Then we may assume that  $q$  is a general point in  $VSP(F, h)$ ,  $W_j$ , and  $W_l$ .*

*Assume moreover that any irreducible component of  $VSP(F, h-1)$  is Rationally Connected and  $\dim VSP(F, h-1) \geq n$  then any irreducible component of  $VSP(F, h)$  is Rationally Connected.*

*Proof.* We have

$$\dim VSP(F, h-1) = n(h-1) + h - 2 - \binom{n+d}{d} + 1 = (h-1)(n+1) - \binom{n+d}{d}.$$

Therefore, under our numerical assumption, this yields

$$(2) \quad \dim VSP(F, h-1) - (n+1) = (n+1)(h-2) - \binom{n+d}{d} \geq 0.$$



Let  $p_1$  and  $p_2$  be two very general points in  $VSP(F, h)$ , with associated decompositions, respectively,

$$\sum_1^h \lambda_i L_i^d \text{ and } \sum_1^h \mu_i G_i^d$$

Let  $q \in VSP(F - \lambda_1 L_1^d, h - 1)$  be a general point with associated decomposition

$$\lambda_1 L_1^d + \sum_2^h \xi_i B_i^d$$

Let  $\nu : Z \rightarrow VSP(F, h)$  be a resolution of singularities. Since  $p_1$  and  $p_2$  are very general we may assume the following:

- ( $\star$ )  $\nu^{-1}(VSP(F - \lambda_1 L_1^d, h - 1))$  and  $\nu^{-1}(VSP(F - \mu_1 G_1^d, h - 1))$  belong to the same irreducible component of  $\text{Hilb}(Z)$ , and  $\nu$  is an isomorphism in a neighbourhood of  $q$ .

Then by construction we have

$$q \in VSP(F - \lambda_1 L_1^d, h - 1) \cap VSP(F - \xi_2 B_2^d, h - 1).$$

Hence by equations (1), (2), and assumption ( $\star$ ) we conclude that

$$VSP(F - \mu_1 G_1^d, h - 1) \cap VSP(F - \xi_2 B_2^d, h - 1) \neq \emptyset.$$

Furthermore the general point of this intersection is a general point of  $VSP(F, h)$ ,  $VSP(F - \mu_1 G_1^d, h - 1)$  and  $VSP(F - \xi_2 B_2^d, h - 1)$ .

To conclude the rational connectedness in case  $\dim VSP(F, h - 1) = n$  consider the variety  $V := \overline{\bigcup_{\lambda} VSP(F - \lambda L_1^d, h - 1)}$ . Then  $V$  has a natural map onto  $\mathbb{P}^1$  with Rationally Connected fibers. Therefore, via the main result of [GHS], we know that  $V$  is a Rationally Connected variety of dimension  $n + 1$ . Then we may argue as before with  $V$  instead of  $VSP(F - \lambda_1 L_1^d, h - 1)$ .  $\square$

Theorem 3.2 allows us to describe birational properties of  $VSP(F, h)$  starting from those of  $VSP_G^{V_{d,n}}(h - 1)$ .

Our next task is to understand how to use  $VSP_G(F', h)$ , with  $F' \in k[x_0, \dots, x_{n-k}]_d$  to study  $VSP_G(F, h)$ . It is difficult, at least to us, to understand it from the algebraic point of view. On the other hand a neat geometric way is at hand. Let  $V_{d,n} \subset \mathbb{P}^N$  be the  $d$ -uple Veronese embedding of  $\mathbb{P}^n$ . Let  $Y \subset V$  be a rational subvariety of dimension  $b$ . Let us think of  $Y$  as the projection of  $V_{\delta,b} \subset \mathbb{P}^{N'}$  for some  $\delta$ .

**Theorem 3.3.** *Let  $Y \subset \mathbb{P}^N$  be a projection of  $V = V_{\delta,b} \subset \mathbb{P}^{N'}$ , and  $p \in \mathbb{P}^N$  a general point. Assume that  $\text{Sec}_{h-1}(V) = \mathbb{P}^{N'}$  and  $h < N - b$ . Then there is an irreducible component  $W \subset VSP_G^Y(h)$  containing general  $h$ -secant linear spaces, and a birational map  $\varphi$  giving rise to the following diagram*

$$\begin{array}{ccc} \text{Sec}_{h-1}(V) \supset \tilde{W} & \xrightarrow{\pi_{h-1|\tilde{W}}} & \mathbb{P}^{N'-N} \\ \vdots \varphi & & \\ \text{Sec}_{h-1}(Y) \supset W & & \end{array}$$

*In particular if  $VSP_G^V(h)$  is Rationally Connected then  $VSP_G^Y(h)$  has a Rationally connected irreducible component of dimension  $N' - 2N + (b + 1)h - 1$ .*

*Remark 3.4.* The hypothesis of Theorem 3.3 are quite strong. In particular the assumption on  $\text{Sec}_{h-1}(V)$  confines its application to finitely many cases. On the other hand we think it is important, at least conceptually, to have a geometric way to “lower the number of variables”.

*Proof.* Let  $\pi : \mathbb{P}^{N'} \dashrightarrow \mathbb{P}^N$  be the projection, with  $\pi(V) = Y$ ,  $p \in \mathbb{P}^N$  a general point, and  $\Pi = \pi^{-1}(p) \cong \mathbb{P}^{N'-N}$ . Since  $p \in \mathbb{P}^N$  is a general point we may assume that the general point of  $\Pi$  is general in  $\mathbb{P}^{N'}$ . Consider  $\pi_{h-1} : \text{Sec}_{h-1}(V) \rightarrow \mathbb{P}^{N'}$  and let  $\tilde{W} \subset \pi_{h-1}^{-1}(\Pi)$  be an irreducible component containing general  $h$ -secant linear spaces. The numerical assumption  $h < N - b$  ensures that:

- i) the general  $h$ -secant linear space to  $V$  does not intersect the center of projection,
- ii) the general  $h$ -secant linear space to  $Y$  intersect  $Y$  in exactly  $h$  points.

Item i) and ii) ensures that the general  $h$ -secant linear space to  $V$  is mapped to a general  $h$ -secant linear space to  $Y$ . This gives the map  $\varphi$ . Item ii) shows that  $\varphi$  is generically injective. Note that the expected dimension of  $\text{Sec}_{h-1}(Y)$  is  $(h+1)b - 1 - N$  while  $\dim \tilde{W} = (h+1)b - 1 - N' + (N' - N)$ . Therefore  $Y$  is not defective and  $W := \varphi(\tilde{W})$  is an irreducible component.

If  $VSP_G^V(h)$  is Rationally connected we may apply the main result of [GHS] to conclude that  $\tilde{W}$  is rationally connected and henceforth  $W$  is rationally connected.  $\square$

As we already noticed hypothesis of Theorem 3.3 are seldom satisfied. The following is a good tool to study Rational Connectedness of  $VSP_G(F, h)$  in many more contests.

**Proposition 3.5.** *For any triple of integers  $(a, b, c)$ , with  $b < n$ , there is a Rationally Connected variety  $W_{a,c,b}^n \subset \text{Hilb}(\mathbb{P}^n)$  with the following properties:*

- a general point in  $W_{a,c,b}^n$  represents a rational subvariety of  $\mathbb{P}^n$  of codimension  $b$ ;
- for any  $Z \subset \mathbb{P}^n \setminus \{(x_0 = \dots = x_{n-b} = 0)\}$  reduced zero dimensional scheme of length  $\leq c$ , there is a Rationally Connected subvariety  $W_{Z,b} \subset W_{a,c,b}^n$ , of dimension at least  $a$ , whose general element  $[Y] \in W_{Z,b}$  represents a rational subvariety of  $\mathbb{P}^n$  of codimension  $b$  containing  $Z$ .

*Proof.* We prove the statement by induction on  $b$ . Assume  $b = 1$ , and consider an equation of the form

$$Y = (x_n A(x_0, \dots, x_{n-1})_{d-1} + B(x_0, \dots, x_{n-1})_d = 0),$$

then, for  $A$  and  $B$  generic,  $Y$  is a rational hypersurface of degree  $d$  with a unique singular point of multiplicity  $d - 1$  at the point  $[0, \dots, 0, 1]$ .

Fix  $d > ac$  and let  $W_{a,c,1}^n$  be the linear span of these hypersurfaces. For any triple  $(a, 1, c)$  and a subset  $Z \subset \mathbb{P}^n \setminus \{[0, \dots, 0, 1]\}$  consider  $W_{Z,1} \subset W_{a,c,1}^n$  as the sublinear system of hypersurfaces containing  $Z$ .

Assume, by induction, that  $W_{a,c,i-1}^n \subset \text{Hilb}(\mathbb{P}^{n-1})$  exist for any  $n$  and  $c$ . Define, for  $i \geq 2$ ,

$$\tilde{W}_{a,c,i}^n := W_{a,c,1}^n \times W_{a,c,i-1}^{n-1} \subset \text{Hilb}(\mathbb{P}^n) \times \text{Hilb}(\mathbb{P}^{n-1}).$$

Let  $[X]$  be a general point in  $W_{a,c,1}^n$ . By construction  $X$  has a point of multiplicity  $d - 1$  at the point  $[0, \dots, 0, 1] \in \mathbb{P}^n$ . Then the projection  $\pi_{[0, \dots, 0, 1]} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$

restricts to a birational map  $\varphi_X : X \dashrightarrow \mathbb{P}^{n-1}$ . Hence we may associate the general element  $([X], [Y]) \in \{[X]\} \times W_{a,c,i-1}^{n-1}$  to the codimension  $i$  subvariety  $\varphi_X^{-1}(Y) \subset \mathbb{P}^n$ . This, see for instance [Ko, Proposition I.6.6.1], yields a rational map

$$\chi : \tilde{W}_{a,c,i}^n \dashrightarrow \text{Hilb}(\mathbb{P}^n).$$

Let  $W_{a,c,i}^n := \overline{\chi(\tilde{W}_{a,c,i}^n)} \subset \text{Hilb}(\mathbb{P}^n)$ . For any  $Z$  we may then define

$$\tilde{W}_{Z,i} := W_{Z,1} \times W_{\pi_{[1,0,\dots,0]}(Z),i-1},$$

and as above  $W_{Z,i} = \overline{\chi(\tilde{W}_{Z,i})}$ .  $\square$

#### 4. RATIONALITY RESULTS

In this section we prove some rationality result for  $VSP$ 's. The first interesting case is that of  $\mathbb{P}^1$ , namely polynomials in two variables. This is probably known but we where not able to find an appropriate reference.

**Theorem 4.1.** *Let  $h > 1$  be a fixed integer. For any integer  $d$  such that*

$$h \leq d \leq 2h - 1,$$

*we have  $VSP(F, h) \cong \mathbb{P}^{2h-d-1}$ .*

*Proof.* We already noticed, see Remark 1.8, that in this case

$$VSP(F, h) \cong VSP_G(F, h).$$

Let  $F$  be a homogeneous polynomial of degree  $d$  and let  $\{[L_1], \dots, [L_h]\}$  be a  $h$ -polar polyhedron of  $F$ , then

$$F = \lambda_1 L_1^d + \dots + \lambda_h L_h^d.$$

We consider the partial derivatives of order  $d - h > 0$  of  $F$ . This partial derivatives are

$$\binom{d-h+1}{d-h} = d - h + 1 \leq h$$

homogeneous polynomials of degree  $h$ .

Let  $X$  be the general normal curve of degree  $h$  in  $\mathbb{P}^h$ . The partial derivatives span a  $(d - h)$ -plane  $H_\partial \subset \mathbb{P}^h$ . The general choice of  $F$  ensures that  $H_\partial \cap X = \emptyset$ . By Remark 1.12 the points  $[L_1^d], \dots, [L_h^d] \in X$  span an hyperplane containing  $H_\partial$ .

The hyperplanes of  $\mathbb{P}^h$  containing  $H_\partial$  are parametrized by  $\mathbb{P}^{2h-d-1}$  and any hyperplane containing  $H_\partial$  intersects  $X$  in a zero dimensional scheme of length  $h$ . This gives rise to an injective morphism

$$\phi : \mathbb{P}^{2h-d-1} \rightarrow VSP(F, h), \Pi \mapsto \Pi \cap X.$$

The varieties  $VSP(F, h)$  and  $\mathbb{P}^{2h-d-1}$  are both smooth by Proposition 1.5 and

$$\dim(VSP(F, h)) = 2h - \binom{d+1}{d} = 2h - d - 1.$$

Hence the injective morphism  $\phi$  is an isomorphism.  $\square$

The next rationality result is for quadratic polynomials. It is well known that two general quadrics can be simultaneously diagonalized. Building on this we can prove the following.

**Theorem 4.2.** *Let  $F \in k[x_0, \dots, x_n]_2$  be a general homogeneous polynomial of degree two. Then  $VSP(F, n + 1)$  is rational.*

*Proof.* Up to a projectivity of  $\mathbb{P}^n$  we may assume that  $F$  is given by

$$F = x_0^2 + \dots + x_n^2.$$

Let  $\Pi$  be a general  $(N - n)$ -plane in  $\mathbb{P}^N = \mathbb{P}(k[x_0, \dots, x_n]_2)$ , and  $[G] \in \Pi$  a general point.

The quadrics  $F$  and  $G$  are general. Then we may assume that the pencil they generate contains exactly  $n + 1$  distinct singular quadric cones, say  $C_0, \dots, C_n$ . Let  $v_i \in \mathbb{P}^n$  the vertex of the cone  $C_i$  for  $i = 0, \dots, n$ . Via the Veronese embedding  $\nu_2 : \mathbb{P}^n \rightarrow \mathbb{P}^N$  we find  $n + 1$  points  $\nu_2(v_i)$  on the Veronese variety  $V_{2,n} \subset \mathbb{P}^N$ .

Let  $A$  be the matrix of  $G$ . Then the cones in the pencil  $\lambda F - G$  are determined by the values of  $\lambda$  such that  $\det(\lambda I - A) = 0$ . In other words the cones  $C_i$  correspond to the eigenvalues of  $A$  and the singular points  $v_i$  are given by the eigenvectors of  $A$ . In particular  $v_i$ 's are linearly independent and in the basis  $\{v_0, \dots, v_n\}$  the matrix  $A$  is diagonal

$$\begin{pmatrix} \lambda_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

We may further assume that  $\{v_0, \dots, v_n\}$  is an orthonormal base. Therefore after the projectivity induced by this change of variables we have that  $F$  is still represented by the identity and  $G$  is diagonal.

Any projectivity of  $\mathbb{P}^n$  induces a projectivity on  $\mathbb{P}^N$  that stabilizes  $V \subset \mathbb{P}^N$ . Hence after the needed projectivities we have

$$\nu_2(v_i) = \nu_2([0, \dots, 0, 1, 0, \dots, 0]) = [x_i^2]$$

Therefore the linear space  $\langle [x_0^2], \dots, [x_n^2] \rangle$  contains both  $[F]$  and  $[G]$ . This construction gives a map

$$\psi : \Pi \dashrightarrow VSP(F, n + 1), [G] \mapsto \{v_0, \dots, v_n\}$$

The birationality of  $\psi$  is immediate once remembered that  $\Pi$  is a codimension  $n$  linear space, and  $\dim(VSP(F, n + 1)) = N - n$ .  $\square$

For conics a bit improvement is at hand.

**Theorem 4.3.** *Let  $F \in k[x_0, x_1, x_2]_2$  be a general homogeneous polynomial of degree two. Then  $VSP(F, 4)$  is birational to the Grassmannian  $\mathbb{G}(1, 4)$ , and hence rational.*

*Proof.* The map is quite simple. The 3-planes passing through  $[F] \in \mathbb{P}^5$  are parametrized by  $\mathbb{G}(1, 4)$  and a general linear space cuts exactly 4 points on the Veronese surface  $V_{2,2} \subset \mathbb{P}^5$ . To conclude it is enough to check that  $\dim VSP(F, 4) = \dim \mathbb{G}(1, 4) = 6$ .  $\square$

We are not able to prove rationality for arbitrary  $n$  and  $h$ . Nonetheless the proof of Theorem 4.2 allow us to prove the following unirationality statement.

**Theorem 4.4.** *Let  $F \in k[x_0, \dots, x_n]_2$  be a general homogeneous polynomial of degree two. Then  $VSP(F, h)$  is unirational.*

*Proof.* We have to prove the statement for  $h > n + 1$ . Let  $\Pi \subset \mathbb{P}^N$  be a codimension  $n$  linear space and  $q \in \Pi$  a point. The proof of Theorem 4.2 shows that for a general

$[F] \in \mathbb{P}^N$  there is a well defined decomposition associated to  $q$ . This can be seen as a rational section

$$\sigma_q : \mathbb{P}^N \dashrightarrow \text{Sec}_n(V_{2,n})$$

We proved that the general fiber of the map  $\pi_n : \text{Sec}_n(V_{2,n}) \rightarrow \mathbb{P}^N$  is rational. Hence we have a well defined birational map

$$\chi : \mathbb{P}^N \times \mathbb{P}^{N-n} \dashrightarrow \text{Sec}_n(V_{2,n}).$$

This means that given a general quadratic polynomial, say  $q$ , and a point in  $\mathbb{P}^{N-n}$  it is well defined an additive decomposition of  $q$  into  $h$  factors. This allows us to define the following map, for  $h > n + 1$

$$\psi_h : \mathbb{P}^{N-n} \times (V_{2,n} \times \mathbb{P}^1)^{h-(n+1)} \dashrightarrow VSP_G(F, h)$$

given by

$$(p, [L_1^2], \lambda_1, \dots, [L_{h-(n+1)}^2], \lambda_{h-(n+1)}) \mapsto (\lambda_1 L_1^2 + \dots + \lambda_{h-(n+1)} L_{h-(n+1)}^2 + \\ + \chi([F - \sum_{i=1}^{h-(n+1)} \lambda_i L_i^2], p)).$$

The map  $\psi_h$  is clearly generically finite, of degree  $\binom{h}{n+1}$ , and dominant. This is enough to show that  $VSP_G(F, h)$  is unirational for  $h > n + 1$ .  $\square$

## 5. RATIONAL CONNECTEDNESS

In this section we prove the result on rational connectedness taking advantage of the preparatory work of the previous sections. In higher degree one cannot expect a result like in the case of quadratic polynomials. It is enough to think of either Mukai Theorem, [Mu1], where is proven that  $VSP(F, 10)$  is a  $K3$  surface for  $F \in k[x_0, x_1, x_2]_6$  general, or Iliev and Ranestad example of a symplectic  $VSP$ , [IR1]. On the other hand we found a nice behaviour for infinitely many degrees and number of variables. Keep in mind that  $VSP(F, h)$  are not empty only for  $h \geq \frac{\binom{n+d}{n}}{n+1}$ .

**Theorem 5.1.** *Assume that for some positive integer  $k < n$  the number  $\frac{\binom{d+n}{n}-1}{k+1}$  is an integer. Then the irreducible components of  $VSP(F, h)$  are Rationally Connected for  $F \in k[x_0, \dots, x_n]_d$  general and  $h \geq \frac{\binom{n+d}{n}-1}{k+1}$ .*

Let us start stating explicitly [Me1, Remark 4.6].

**Proposition 5.2.** *Let  $V_{\delta,n} \subset \mathbb{P}^N$  be a Veronese embedding, for  $\delta \geq 4$ . Assume that  $\text{cod Sec}_h(V) \geq n + 1$ . Then through a general point of  $\text{Sec}_h(V)$  there is a unique  $h$ -linear space  $(h + 1)$ -secant to  $V$ .*

*Proof.* Let  $z \in \text{Sec}_h(V)$  be a general point. Assume that  $\langle p_0, \dots, p_h \rangle \ni z$  and  $z \in \langle q_0, \dots, q_h \rangle$  for  $h$ -tuple of points in  $V$ . Then Terracini Lemma, [CC], yields

$$\mathbb{T}_z \text{Sec}_h(V) = \langle \mathbb{T}_{q_0} V, \dots, \mathbb{T}_{q_h} V \rangle = \langle \mathbb{T}_{p_0} V, \dots, \mathbb{T}_{p_h} V \rangle.$$

Therefore the general hyperplane section  $H \cap V$  singular at  $\{p_0, \dots, p_h\}$  is singular at  $\{q_0, \dots, q_h\}$  as well. On the other hand, by [Me1, Corollary 4.5],  $V$  is not  $h$ -weakly defective. Then by [CC, Theorem 1.4] the general hyperplane section  $H \cap V$  tangent at  $h$ -general points  $\{p_0, \dots, p_h\}$ , of  $V$  is singular only at those points. This gives  $\{p_0, \dots, p_h\} = \{q_0, \dots, q_h\}$  and proves the proposition.  $\square$

*Proof of Theorem 5.1.* Without loss of generality, to simplify notation, we may assume that  $VSP_G(F, h)$  is irreducible. Fix  $h = \frac{\binom{n+d}{n}-1}{k+1} = \frac{N}{k+1}$ , and assume that  $[\Lambda_x], [\Lambda_y] \in VSP_G(F, h)$  are two general points, with  $\Lambda_x = \langle x_1, \dots, x_h \rangle$  and  $\Lambda_y = \langle y_1, \dots, y_h \rangle$ .

In the notation of Proposition 3.5, let  $W_1 := W_{h, 2h, n-k}^n$ . Let  $[X] \in W_1$  be a general element, then the numerical assumption, together with [AH] main Theorem, yields

$$\dim \text{Sec}_{h-1}(X) = h(k+1) - 1 = N - 1.$$

Then  $\text{Sec}_{h-1}(X) \subset \mathbb{P}^N$  is an hypersurface of degree say  $\alpha$ .

Remark 1.10 allow us to define a rational map as follows

$$\varphi : W_1 \dashrightarrow \mathbb{P}(k[x_0, \dots, x_N]_\alpha)$$

defined sending  $X$  to its  $h$ -secant. Let  $SW_1 = \overline{\varphi(W_1)}$  and  $SW_{1[F]} \subset SW_1$  be an irreducible component of maximal dimension of the subvariety parametrizing elements of  $SW_1$  passing through a point  $[F] \in \mathbb{P}^N$ . The variety  $X$  is rational. Hence it is the projection of a  $\delta$ -Veronese embedding of  $\mathbb{P}^k$ , for some  $\delta \gg 0$ . Without loss of generality we may assume that  $\text{Sec}_{h-1}(X)$  is the projection of the  $(h-1)$ -secant variety of the Veronese embedding (otherwise we restrict to this irreducible component). Then by Proposition 5.2 there is a unique  $h$ -secant linear space to  $X$  through a general point of  $\text{Sec}_{h-1}(X)$ .

We may then define a rational dominant map

$$(3) \quad \psi : SW_{1[F]} \dashrightarrow VSP_G(F, h) \subset \mathbb{G}(h-1, N)$$

sending a general secant in  $SW_{1[F]}$  to the unique  $h$ -secant linear space passing through  $[F] \in \mathbb{P}^N$ . In the notation of Proposition 3.5 we have

$$\overline{\psi^{-1}([\Lambda_x])} \supseteq \varphi(W_{\{x_1, \dots, x_h\}, n-k}),$$

$$\overline{\psi^{-1}([\Lambda_y])} \supseteq \varphi(W_{\{y_1, \dots, y_h\}, n-k}),$$

and

$$\overline{\psi^{-1}([\Lambda_x])} \cap \overline{\psi^{-1}([\Lambda_y])} \supseteq \varphi(W_{\{x_1, \dots, x_h, y_1, \dots, y_h\}, n-k}).$$

The subvarieties  $W_{\{x_1, \dots, x_h\}, n-k}$  and  $W_{\{y_1, \dots, y_h\}, n-k}$  are Rationally Connected. Therefore  $SW_{1[F]}$  is rationally chain connected by two rational curves intersecting in a general point of  $\varphi(W_{\{x_1, \dots, x_h, y_1, \dots, y_h\}, n-k})$ .

*Claim 1.* The variety  $SW_{1[F]}$  is Rationally Connected.

*Proof of the Claim.* The variety  $SW_1 \subset \mathbb{P}(k[x_0, \dots, x_N]_\alpha)$  parametrizes divisors in  $\mathbb{P}^N$ . Let  $H_{[F]} \subset \mathbb{P}(k[x_0, \dots, x_N]_\alpha)$  be the hyperplane parametrizing the hypersurfaces passing through  $[F]$ . Then we have  $SW_{1[F]} = SW_1 \cap H_{[F]}$ .

A general point  $[T] \in SW_1$  represents a projection of the secant variety to a Veronese. This yields that  $T$  is singular in codimension 1. Moreover a general point of  $\text{Sing}(T)$  is a double point. That is, by Proposition 5.2, for  $t \in \text{Sing}(T)$  general point there are two linear spaces  $h$ -secant to the Veronese passing through  $t$ . We can therefore assume that the general point  $x \in \varphi(W_{\{x_1, \dots, x_h, y_1, \dots, y_h\}, n-k})$  is a general point of  $SW_1$ .

Let  $\Sigma_{[F]} \subset SW_{1[F]}$  be the subvariety parametrizing secant varieties with more than one  $(h-1)$ -linear space  $h$ -secant passing through  $[F]$ . First we compute the codimension of  $\Sigma_{[F]}$ . We already observed that for  $[T] \in SW_1$  the hypersurface  $T$

is singular along a codimension 1 set. Therefore the set of hypersurfaces singular at a general point  $[F] \in \mathbb{P}^N$  is in codimension 2 in  $SW_1$ ,

$$\text{cod}_{SW_1} \Sigma_{[F]} = 2.$$

All these hypersurfaces are clearly contained in  $SW_{1[F]}$ , therefore we conclude that

$$\text{cod}_{SW_{1[F]}} \Sigma_{[F]} = 1.$$

The above construction shows that  $SW_{1[F]}$  is rationally chain connected by chains of rational curves passing through general points of  $\Sigma_{[F]}$ . If the general point of  $\Sigma_{[F]}$  is smooth in  $SW_{1[F]}$  then the claim is proved.

Assume that  $SW_{1[F]}$  is singular along  $\Sigma_{[F]}$ . Let  $\nu : Z \rightarrow SW_{1[F]}$  be the normalization, and

$$S_{\{x_i\}\{y_j\}} := \varphi(W_{\{x_1, \dots, x_h\}, n-k}) \cap \varphi(W_{\{y_1, \dots, y_h\}, n-k})$$

be the intersection. By construction we have the following:

- (a)  $\dim S_{\{x_i\}\{y_j\}} \geq h$ ;
- (b) for a general  $s \in \Sigma_{[F]}$  there are exactly two  $h$ -tuples such that  $S_{\{x_i^s\}\{y_j^s\}} \ni s$ .

Let us consider  $\Sigma_{[F]}$  with its complex topology. The morphism  $\nu$  is a finite covering outside a codimension 1 set, say  $K$ . For any point  $s \in K^c$  there is an open neighborhood (in the complex topology), say  $B_s$ , such that  $\nu|_{\nu^{-1}(B_s)}$  is finite and étale. The set  $K$  is closed and of measure zero. That is for any  $\epsilon > 0$  there is an open  $V \subset \Sigma_{[F]}$  such that  $V \supset K$  and  $V$  has measure bounded by  $\epsilon$ . Then  $V^c$  is a compact space and we may cover it with finitely many open sets  $\{B_{s_i}\}_{i=1, \dots, m}$ . Then, by (a) and (b) above, we may assume that for two general decompositions

$$\dim(\varphi(W_{\{x_1, \dots, x_h\}, n-k}) \cap \varphi(W_{\{y_1, \dots, y_h\}, n-k}) \cap V^c) > 0.$$

By construction  $\nu|_{\nu^{-1}(B_s)}$  is finite and étale. The compact  $V^c$  is covered by a finite number of  $B_s$ . This, going back to Zariski topology and keeping in mind (a), together with the rational connectedness of the subvarieties  $W$ , shows that the  $MRC$  fibration of  $Z$  has at most finitely many fibers. Hence the irreducible varieties  $Z$  and  $SW_{1[F]}$  are Rationally Connected.  $\square$

The claim shows that  $SW_{1[F]}$  is Rationally Connected and hence  $VSP_G(F, h)$  is Rationally Connected, via the map  $\psi$  of equation (3). To conclude the proof for  $h > \frac{n+d}{k+1}$  it is then enough to apply Theorem 3.2.  $\square$

For special values a more precise statement can be obtained.

**Theorem 5.3.** *The variety  $VSP(F, h)$  is Rationally Connected in the following cases:*

- a)  $F \in k[x_0, x_1, x_2]_4$  and  $h \geq 6$ ,
- c)  $F \in k[x_0, \dots, x_4]_3$  and  $h \geq 8$ ,
- b)  $F \in k[x_0, \dots, x_3]_3$  and  $h \geq 6$ ,
- d)  $F \in k[x_0, x_1, x_2]_3$  and  $h \geq 4$ ,

*The variety  $VSP(F, h)$  is uniruled for  $F \in k[x_0, \dots, x_4]_3$  and  $h \geq 7$ .*

*Proof.* In cases a) and b) we know that  $VSP(F, 6)$ , [Mu1], and  $VSP(F, 8)$ , [RS], respectively are rational of dimension  $n+1$ . Then to conclude it is enough to apply Theorem 3.2.

In case c) observe that there is a twisted cubic in  $\mathbb{P}^3$  through 6 points. Then Theorems 4.1 and 3.3 produce a chain of  $\mathbb{P}^2$  through very general points of  $VSP(F, 6)$ . Then we apply Theorem 3.2 to conclude for arbitrary  $h \geq 7$ . In case d) we have  $\mathbb{P}^2 \cong VSP(F, 4)$  and we conclude again by Theorem 3.2.

Finally observe that there is a rational quartic in  $\mathbb{P}^4$  through 7 points. Then Theorems 4.1 and 3.3 produce a  $\mathbb{P}^1$  through a general point of  $VSP(F, h)$ , for  $h \geq 7$ .  $\square$

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